Centers, centralizers, and normalizers

Suppose G is a group, and let $S \subseteq G$.

- The <u>center</u> of G is

$$Z(G) = \{g \in G : \forall h \in G,\ gh = hg\}$$

- The <u>centralizer of S in G</u> is

$$C_G(S) = \{g \in G : \forall s \in S,\ gs = sg\}.$$

- The <u>normalizer of S in G</u> is

$$N_G(S) = \{g \in G : gSg^{-1} = S\}.$$

First observations:

- $\forall S \subseteq G$, $Z(G) \subseteq C_G(S)$.
- $\forall S \subseteq G$, $C_G(S) \subseteq N_G(S)$.

  If $g \in C_G(S)$ then, $\forall s \in S$, $gs = sg \Rightarrow gsg^{-1} = s$.

  Therefore $gSg^{-1} = \{\overset{=s}{gsg^{-1}} : s \in S\} = S$.

- $Z(G) = C_G(G)$.
- $e \in Z(G)$. $\forall g \in G$, $eg = ge = g$.
- If $H \leq G$ then $H \trianglelefteq G \Leftrightarrow N_G(H) = G$.
- If G is Abelian then $Z(G) = C_G(S) = N_G(S) = G$.

Thm: For any group G and $S \subseteq G$, $Z(G) \leq C_G(S) \leq N_G(S) \leq G$.

Pf: • $N_G(S) \leq G$: (subgroup crit.)

• $e \in Z(G) \Rightarrow e \in N_G(S)$, so $N_G(S) \neq \phi$. (non-empty)

• If $g,h \in N_G(S)$ then

$$(gh)S(gh)^{-1} = \{(gh)s(gh)^{-1} : s \in S\}$$
$$= \{g(hsh^{-1})g^{-1} : s \in S\}$$

(closed under mult.)

$$= gSg^{-1} = S \Rightarrow gh \in N_G(S).$$

• Suppose $g \in N_G(S)$. Then $gSg^{-1} = S$, so:

- $\forall s \in S$, $gsg^{-1} = s'$ for some $s' \in S$

$\Rightarrow g^{-1}s'g = s \Rightarrow s \in g^{-1}Sg$.

So $S \subseteq g^{-1}Sg$.

- $\forall s \in S$, $\exists s' \in S$ s.t. $gs'g^{-1} = s$

$\Rightarrow g^{-1}sg = s' \in S$.

Therefore $g^{-1}Sg \subseteq S$.

(closed under inverses)

So $g^{-1}Sg = S$, which means that $g^{-1} \in N_G(S)$.

Therefore, $N_G(S) \leq G$.

- $C_G(S) \leq N_G(S)$: Already know that $C_G(S) \subseteq N_G(S)$, so just need to show that $C_G(S)$ is a group.

  - $e \in C_G(S) \Rightarrow C_G(S) \neq \emptyset$.

  - If $g, h \in C_G(S)$ then $\forall s \in S$,

    $$(gh)s = g(hs) \underset{h \in C_G(S)}{=} g(sh) = (gs)h \underset{g \in C_G(S)}{=} (sg)h = s(gh)$$

    $\Rightarrow gh \in C_G(S)$.

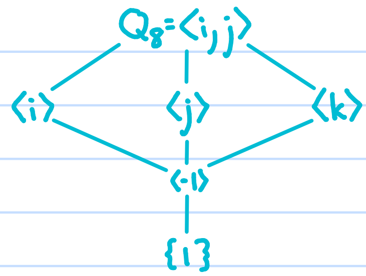
  - If $g \in C_G(S)$ then $\forall s \in S$,

    $$gs = sg \Rightarrow s = g^{-1}sg \Rightarrow sg^{-1} = g^{-1}s \Rightarrow g^{-1} \in C_G(S).$$

  Therefore, $C_G(S) \leq N_G(S)$.

- $Z(G) \leq C_G(S)$: $Z(G) \subseteq C_G(S)$ and $Z(G) = C_G(G) \leq G \Rightarrow Z(G) \leq C_G(S)$. $\blacksquare$

Ex: $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$



- $Z(Q_8) = \{\pm 1\}$

  $(-1)h = h(-1), \ \forall h \in Q_8 \Rightarrow -1 \in Z(Q_8)$

  $ij \neq ji, \ ik \neq ki \Rightarrow \pm i, \pm j, \pm k \notin Z(Q_8)$

- $C_{Q_8}(\langle i \rangle) = \langle i \rangle$

  $i \cdot i^n = i^n \cdot i, \ \forall n \in \mathbb{Z} \Rightarrow i \in C_{Q_8}(\langle i \rangle) \Rightarrow \langle i \rangle \subseteq C_{Q_8}(\langle i \rangle)$

  $\pm j, \pm k \notin C_{Q_8}(\langle i \rangle)$

- $N_{Q_8}(\langle i \rangle) = Q_8$

  $\langle i \rangle \trianglelefteq Q_8$ (previous video)

- $C_{Q_8}(\{i, j\}) = \{\pm 1\}$

  $Z(Q_8) \subseteq C_{Q_8}(\{i, j\}) \Rightarrow \pm 1 \in C_{Q_8}(\{i, j\})$

  $ij \neq ji, \ ki \neq ik \Rightarrow \pm i, \pm j, \pm k \notin C_{Q_8}(\{i, j\})$

- $N_{Q_8}(\{i, j\}) = \{\pm 1\}$

  $C_{Q_8}(\{i, j\}) \subseteq N_{Q_8}(\{i, j\}) \Rightarrow \pm 1 \in N_{Q_8}(\{i, j\})$

  $i j i^{-1} = (\overbrace{ij}^{=k})(-i) = -ki = -j \notin \{i, j\} \Rightarrow \pm i \notin N_{Q_8}(\{i, j\})$

  $j i j^{-1} = (\overbrace{ji}^{=-k})(-j) = kj = -i \notin \{i, j\} \Rightarrow \pm j \notin N_{Q_8}(\{i, j\})$

  $k i k^{-1} = (ki)(-k) = -jk = -i \notin \{i, j\} \Rightarrow \pm k \notin N_{Q_8}(\{i, j\})$